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PATHS AND CYCLES IN TOURNAMENTS

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ABSTRACT. Sufficient conditions are given for the existence of an oriented path with given end vertices in a tournament. As a consequence a conjecture of Rosenfeld is established. This states that if n is large enough, then every non-strongly oriented cycle of order n is contained in every tournament of order n.

It is well known and easy to see that every tournament has a directed hamilton path. Rosenfeld [**8**] conjectured that if n is large enough, then any oriented path of order n is contained in any tournament of order n. This has been established for alternating paths by Grünbaum [**5**] and Rosenfeld [**8**], for paths with two blocks (a block being a maximal directed subpath) by Alspach and Rosenfeld [**1**] and Straight [**10**], for paths where the ith block has length at least $i+1$ by Alspach and Rosenfeld [**1**] and, curiously, for all paths if n is a power of 2 by Forcade [**4**]. Reid and Wormald [**7**] have shown that every oriented path of order n is contained in every tournament of order $3n/2$.

It is easy to show that a tournament has a strongly oriented hamilton cycle if and only if it is strongly connected. Rosenfeld in [**9**] conjectured that any non-strongly oriented cycle of order n is contained in any tournament of order n, provided n is large enough. This has been verified for cycles with a block of length $n-1$ by Grünbaum, for alternating cycles by Rosenfeld [**9**] and Thomassen [**11**], and for cycles with just two blocks by Benhocine and Wojda [**2**]. (It has also been shown by Heydemann, Sotteau and Thomassen [**6**] that every digraph of order n with $(n-1)(n-2)+3$ edges contains every non-strong oriented cycle of order n.)

In this paper we prove both conjectures (the first is of course a consequence of the second). "Large enough" in this case means at least 2^{128}, or about 10^{39}. In fact it seems that the path conjecture is true for $n \geq 8$ (indeed there are probably just three pairs (P, T) with $P \not\subset T$) and that the cycle conjecture is true for $n \geq 9$. We stress that we make no attempt to give a small lower bound, but aim to establish the conjecture in the shortest time possible. The main result is Theorem 14, which rests on Lemmas 9, 11 and 13. Roughly speaking, Lemma 13 proves the conjecture if the cycle has two separate and fair sized blocks (as do most cycles), Lemma 9 proves it if the cycle has a huge block (as in Grünbaum's result) and Lemma 11 takes care of cycles with many small blocks (such as alternating cycles).

The proof of the conjecture is in the third section. In the first two sections we establish results of independent interest concerning the existence of oriented paths with specified end vertices. Apart from the odd detail, they are as follows: let P

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be an oriented path of order n with end blocks of lengths k and l, and let T be a tournament containing subsets of vertices K and L with $|K| = k+1$ and $|L| = l+1$. Theorem 1 states that if T has order $n+1$ then we can find P in T with initial vertex in K, and Theorem 5 states that if T has order $n+2$ then we can find P in T with initial vertex in K and end vertex in L. These theorems are true for all n and are proved by induction. They are best possible in that the values given for $|K|$, $|L|$ and $|T|$ cannot be reduced. Theorem 1 is proved even though it is largely superseded by Theorem 5, because the proof contains all the ideas of the proof of Theorem 5 but needs much less time spent on small cases. The proof of Theorem 14 (the cycle conjecture) rests heavily on Theorem 5.

The notation is largely that of Bollobás [**3**]. If T is a tournament and u is a vertex of T we write $u \in T$ instead of $u \in V(T)$. If v is another vertex of T we indicate the edge directed from u to v by uv, and $uv \in T$ is written rather than $uv \in E(T)$. The transitive tournament with vertex set $\{x_1, \ldots, x_k\}$ is denoted by TT_k or $\langle x_1, \ldots, x_k \rangle$, and has all edges $x_i x_j$ with $i < j$. Note that $TT_k \subset T$ if $|T| \geq 2^{k-1}$. If $u \in T$ we write

$$D_T^+(u) = \{v \in T; uv \in T\} \quad \text{and} \quad D_T^-(u) = \{v \in T; vu \in T\},$$

with $d_T^+(u) = |D_T^+(u)|$ and $d_T^-(u) = |D_T^-(u)|$. The subscripts are sometimes omitted. If X and Y are disjoint subsets of $V(T)$ and $xy \in T$ for all $x \in X$ and $y \in Y$ we write $X \Rightarrow Y$ or $Y \Leftarrow X$. If $X = \{x\}$ this becomes $x \Rightarrow Y$ or $Y \Leftarrow x$. Note $x \Rightarrow Y$ implies $x \notin Y$.

Some particular tournaments used are TC_3, TC_5 and T_4^i, $1 \leq i \leq 4$. TC_n has vertices $x_1, \ldots, x_n$ with $x_i x_{i+1} \in TC_n$ (addition mod n) and $x_i x_{i+2} \in TC_5$ for all i. The tournaments T_4^i are the four tournaments of order four: T_4^1 is TT_4, T_4^2 is $x \Rightarrow TC_3$ and T_4^3 is $x \Leftarrow TC_3$. T_4^4 is the strong tournament of order 4, having vertices a, b, c, d and edges ab, bc, cd, da, ac and bd.

Given an oriented path P which starts with k forward edges, followed by l backward edges and m forward edges, we may write $P = P(+k, l, m, \ldots)$. The symbol $P(-k, l, m, \ldots)$ denotes a path beginning with k backward edges, then l forward edges and m backward edges. If the sign is omitted it is assumed positive. $P(k)$ therefore denotes a directed path of order $k+1$ and length k; it has one block. The first and last vertices of P are the initial and end vertices.

All tournaments contain a directed hamilton path, and all oriented paths are contained in the transitive tournament of the same order. If X and Y are disjoint subsets of the vertices of a tournament, the notation $X^+ \to Y^+$ denotes a path $P(|X| + |Y| - 1)$ consisting of a hamilton path forwards through X, an edge to Y and a hamilton path in Y. Reversing the signs reverses the hamilton paths; thus $X^+ \to Y^-$ denotes $P(|X|, |Y|-1)$ and $X^- \to Y^-$ denotes $P(-(|X|-1),\ 1, |Y|-1)$. The path $X^+ \to Y^+$ may be written $x \to Y^+$ if $X = \{x\}$. The expression $u \to W^+ \to v$ does not imply $W \neq \emptyset$, and would denote the edge uv if W were empty. Finally, if $x \in X$ and we wish to specify a hamilton path beginning at x we write xX^+; likewise xX^+y is a path through X from x to $y \in X$. Thus $zX^+ \leftarrow w$ represents $P(|X| - 1, 1)$ beginning at $z \in X$ and ending at $w \notin X$.

1. Paths with specified initial vertex. Given an oriented path P and tournament T with $|T| \geq |P|$, it would be helpful to have conditions on a subset $K \subset T$ showing when we may find a copy of P in T whose initial vertex is in K. Clearly if K is too small this is hopeless; indeed if the length of the first block in P is k, $|K| \leq k$ and $K \Leftarrow (T-K)$, there is no copy of P in T starting in K. It turns out that if $|K| \geq k+1$ it is usually possible to find P beginning in K, and moreover if $|T| \geq |P|+1$ it is always possible, as the next theorem shows.

THEOREM 1. *Let P be an oriented path of order n and first block length k. Let T be a tournament of order $n+1$ and let K be a set of at least $k+1$ vertices of T. Then there is a copy of P in T with initial vertex in K.*

PROOF. The proof is by induction on n. Let $P = P(k, m, p, \ldots)$. If $P = P(k)$, then the theorem is immediate; this observation provides the base of the induction. From now on we assume $m \geq 1$. Observe that we are done by the induction hypothesis if we can find a path P^* in T with initial vertex in K and end vertex of indegree or outdegree (as appropriate) in $T - P^*$ at least $k'+1$, such that P is equivalent to P^* followed by an edge (of the appropriate direction) followed by a path with first block length k'.

Case (a): $k \geq 2$. Suppose there is a $w \in K$ with $d_T^+(w) \geq 2$. Then we may choose $X \subset D_K^-(w)$ with $|X| = \min(d_K^-(w), k-2)$ and let $P^* = X^+ \to w$.

Otherwise $d_T^+(w) \leq 1$ for every $w \in K$. Then $k = 2$, $K = TC_3$ and $K \Leftarrow (T-K)$. By the induction hypothesis $T - K$ contains a copy of $P' = P(-(m-1), p, \ldots)$, giving the path $P = K^+ \leftarrow P'$ beginning in K.

Case (b): $k = 1$, $m \geq 4$. We may suppose K is the edge uv, and let $H = T - K$. Then $|H| = n - 1 \geq k + m \geq 5$, so there is a vertex $w \in H$ with $d_H^-(w) \geq 2$. Let z be the end vertex of a directed hamilton path in $D_H^+(w)$ (put $z = w$ if $d_H^+(w) = 0$); put $\delta = 1$ if $zv \in T$ and $\delta = 0$ if $vz \in T$. Let X be a subset of $D_H^+(w)$ containing a path of order $|X|$ ending at z, with

$$|X| = \min(d_H^+(w), m - 2 - \delta).$$

Thus we have

$$\begin{array}{lll} \text{either} & P^* = u \to v \leftarrow zX^- \leftarrow w & \text{if } zv \in T, \\ \text{or} & P^* = v \to zX^- \leftarrow w & \text{if } vz \in T. \end{array}$$

Case (c): $k = 1$, $m = 3$. Again let K be the edge uv and let $H = T - K$. If $w \in H$ has $d_H^-(w) \geq 3$, we have $P = v \to w \leftarrow P(-2, p, \ldots)$ if $vw \in T$ or $P = u \to v \leftarrow w \leftarrow P(-1, p, \ldots)$ if $wv \in T$. Otherwise $d_H^-(w) \leq 2$ for all $w \in H$, so $|H| \leq 5$, $n = 5$ or 6, and H is TC_5, T_4^2 or T_4^4. In any of these tournaments we can find at least three vertices which are the initial vertices of copies of $P(-3, p)$, where $p = 0$ or 1 as appropriate. If any of these three vertices is in $D_H^+(v)$ we are home. Otherwise $d_H^-(v) \geq 3$, and so we get $u \to v \leftarrow P(-2, p)$.

Case (d): $k = 1$, $m = 2$. With K and H as before, we find $u \to v \leftarrow P(-1, p, \ldots)$ if $d_H^-(v) \geq 2$ or $v \to P(-2, p, \ldots)$ if $d_H^+(v) \geq 3$. If neither of these holds then $d_H^-(v) \leq 1$ and $d_H^+(v) \leq 2$, so $|H| \leq 3$. Hence $n = 4$, $P = P(1, 2)$, and the result is verified by inspection.

Case (e): $k = m = 1$. In this case, if $d_H^+(v) \geq 2$ we have $v \to P(-1, p, \ldots)$. Otherwise $d_H^-(v) \geq n - 2 \geq p + 1$, giving $u \to v \leftarrow P(p, \ldots)$. □

The following corollary is immediate.

COROLLARY 2. *Every oriented path of order n is contained in every tournament of order $n+1$.*

We will later (Theorem 14) give a proof that every oriented path of order n is contained in every tournament of order n, provided $n \geq n_0$. The value of n_0 we will give however is large compared to the likely best possible value of n_0, namely 8. The following assertion, if true, would give this low value for n_0:

Given an oriented path P of order n with first block length k, a tournament T of order n and $K \subset T, |K| = k+2$, there is a copy of P in T with initial vertex in K unless $n < 8$ or $P = P(1, n-2)$, $K = TC_3$ and $K \Leftarrow (T-K)$. It should be possible to prove this assertion using the method of proof of Theorem 1, but this would involve a lot of analysis of small cases which would best be done on a computer.

2. Paths with both ends specified. It is clear from the proof of Theorem 1 that there is an enormous advantage in being able to specify the initial vertex of a path. It is natural to try and go a step further and specify the end vertex also. An obvious extension of Theorem 1 would be to require the path P to have initial vertex in K and end vertex in L, where $|L| = l+1$ and l is the length of the final block of P. This is achieved in Theorems 3–5 but at the expense of having a further spare vertex in the tournament. Nonetheless Theorems 3–5 will provide the main tool in the proof of the principal result of §3.

We begin with two theorems dealing with cases not covered by Theorem 5. The proofs of Theorems 4 and 5 follow very closely that of Theorem 1.

THEOREM 3. *Let P be an oriented path of order n with just two blocks, of lengths k and $l = n-1-k$. Let T be a tournament of order $n+1$ and let K, L be disjoint subsets of $V(T)$ of orders $k+1$ and $l+1$ respectively. Then there is a copy of P with initial vertex in K and end vertex in L.*

PROOF. Choose a hamilton path in K ending at $u \in K$ and a hamilton path in L ending at $v \in L$. These two paths together with the edge uv or vu contain a suitable copy of P. □

THEOREM 4. *Let P be an oriented path of order n with just three blocks, of lengths $k, 1$ and $l = n-2-k$. Let T be a tournament and let K, L be disjoint subsets of $V(T)$ with $|L| \geq l+1$. Then there is a copy of P in T with initial vertex in K and end vertex in L if*

either (i) $|T| = n+1$, $|K| \geq k+2$ *and* $l \geq 2$,
or (ii) $|T| = n+2$, $|K| = k+3$ *and* $l = 1$.

PROOF. The proof is by induction on n. Let $\delta = 0$ if $l \geq 2$ and $\delta = 1$ if $l = 1$.

Case (a): $k \geq 2$. Suppose there is a vertex $w \in K$ with $d_K^+(w) \geq 3+\delta$. Choose $X \subset D_K^-(w)$ with $|X| = \min(d_K^-(w), k-2)$. A similar argument to that used in the proof of Theorem 1 shows that, by the induction hypothesis, there is a copy of P beginning with $X^+ \rightarrow w$ and ending in L.

Otherwise every vertex $w \in K$ has $d_K^+(w) \leq 2+\delta$, and so $d_K^-(w) \geq k-1$. Since $|K| \geq 4$ we may choose $w \in K$ with $d_K^+(w) \geq 2$ and $d_K^-(w) \geq k-1$. Choose $X \subset D_K^-(w)$ with $|X| = k-1$. By Theorem 3 there is a copy of P beginning with $X^+ \rightarrow w$ and ending in L.

Case (b): $k = 1$. If $|K| \geq 4$ or $K = TT_3$ there is a $w \in K$ with $d_K^+(w) \geq 2$, so by Theorem 3 there is a copy of P beginning at w and ending in L.

Otherwise $l \geq 2$ and $K = TC_3$. Choose a hamilton path in L and let the first and second vertices of it be u and v respectively. Let the vertices of K be a, b, c with ab, bc and ca in T. If $ua \in T$ we have $c \to a \leftarrow u \to vP(l-1)$. So we assume $au \in T$ and likewise $bu, cu \in T$. If $av \in T$ we get $c \to u \leftarrow a \to vP(l-1)$, so we assume $va \in T$ and likewise $vb, vc \in T$. Finally, either $ux \in T$ for all $x \in L$, in which case we have $c \to a \leftarrow v \to b \to uP(l-2)$, or else $yu \in T$ for some $y \in L$. In the latter case it is easily seen that there is a path P' of length l in L beginning at v (or see Lemma 7), giving $P = b \to c \leftarrow vP'$. $\square$

The main result of this section now follows.

THEOREM 5. *Let P be an oriented path of order n with first block length k and last block length l, where $k + l \leq n - 3$. Let T be a tournament of order $n + 2$ and let K, L be disjoint subsets of T of orders at least $k + 1$ and $l + 1$ respectively. Then there is a copy of P in T with initial vertex in K and end vertex in L.*

PROOF. The proof is by induction on n. Let $P = P(k, m, \ldots, p, l)$.

Case (a): $k \geq 2$ *or* $l \geq 2$. Without loss of generality assume $k \geq 2$. Let $H = T - L$. Suppose there is a vertex $w \in K$ with $d_H^+(w) \geq 2$. Choose $X \subset D_K^-(w)$ with $|X| = \min(k-2, d_k^-(w))$. By the induction hypothesis there is a copy of P beginning with $X^+ \to w$ and ending in L.

Otherwise $d_H^+(w) \leq 1$ for every $w \in K$. Then $K = TC_3$ and $K \Leftarrow (T - K - L)$. Again by the induction hypothesis or by Theorems 3 and 4 there is a copy of $P' = P(-(m-1), \ldots, p, l)$ in $T' = T - K$ beginning in $T' - L$ and ending in L (note P' has at least two blocks since $k + l \leq n - 3$). Since $K \Leftarrow (T' - L)$ we have the copy $K^+ \leftarrow P'$ of P we desire.

For the remaining cases, in which $k = l = 1$, we assume $K = uv$, $L = xy$ and $H = T - K - L$.

Case (b): $k = l = 1$, $m \geq 4$. Let $K = uv$, and choose $w \in H$ with $d_H^-(w) \geq 2$; this can be done as $|H| = n - k - l \geq m + 1 \geq 5$. Pick a hamilton path in $D_H^+(w)$ and let z be its end vertex. (Put $z = w$ if $d_H^+(w) = 0$.) Let X be the set of the last $\min(m - 2 - \delta, d_H^+(w))$ vertices of this path, where $\delta = 1$ if $zv \in T$ and $\delta = 0$ if $vz \in T$. Observe that $z \in X$ if $X \neq \emptyset$. Then by Theorem 3 or by the induction hypothesis there is a copy of P ending in L and beginning

$$\begin{array}{ll} u \to v \leftarrow zX^- \leftarrow w & \text{if } zv \in T, \\ \text{or} \quad v \to zX^- \leftarrow w & \text{if } vz \in T, \end{array}$$

unless $P = P(1, m, 1, 1)$. This case is identical to the case $P = P(1, 1, p, 1)$ and is covered by Case (c).

Case (c): $k = l = 1$, $m \leq 3$, $n \geq 7$. Suppose $m \geq 2$, and consider first the case $P \neq P(1, m, 1, 1)$. If $d_H^+(v) \geq m + 1$ or if $d_H^-(v) \geq m$, arguments of a by now familiar kind show we are home. Otherwise $|H| = n - 2 \leq 2m - 1 \leq 5$, so $n = 7$ and $m = 3$. But this means $P = P(1, 3, 1, 1)$. So suppose now $P = P(1, m, 1, 1)$. We are again home if $d_H^+(y) \geq 2$ or $d_H^-(y) \geq m + 1$. But one of these must happen, for otherwise $|H| = n - 2 \leq m + 1 \leq 4$. Thus the case $m \geq 2$ is taken care of, and $p \geq 2$ is dealt with similarly.

This leaves $P = P(1,1,s,\ldots,1,1)$, where $s \geq 1$. If $d_H^+(v) \geq 2$ we are home by induction, so we may assume $d_H^+(v) \leq 1$ and hence $d_H^-(v) \geq n-3$. Here again we are home by induction unless $P = P(1,1,n-5,1,1)$. But then either $d_H^-(x) \geq 2$, in which case we may apply induction, or $d_H^+(x) \geq n-3$, when choosing $X \subset D_H^-(v) \cap D_H^+(x)$ with $|X| = n-4$ gives $P = u \to v \leftarrow X^+ \leftarrow x \to y$.

Case (d): $k = l = 1$, $n \leq 6$. The cases $P = P(1,2,1,1)$ and $P = P(1,1,1,1,1)$ were in fact covered by the proof of Case (c). This leaves three possibilities. Since the whole proof is based on induction the small cases are not insignificant, and we outline them here.

$P = P(1,3,1)$: Suppose $w \in H$ has $d_H^-(w) \geq 2$. Then by Theorem 3 we can find $P' = P(-1,1)$ from $D_H^-(w)$ to L, giving $P = u \to v \leftarrow w \leftarrow P'$ unless $vw \in T$. If further $d_H^-(w) = 3$ we now find $P' = P(-2,1)$ from $D_H^-(w)$ to L, giving $P = v \to w \leftarrow P'$ from K to L. Thus we may suppose $d_H^-(w) \leq 2$ for all $w \in H$, and that $d_H^-(w) = 2$ implies $vw \in T$. Due to the symmetry of P, a similar argument shows we may suppose $d_H^+(w) \leq 2$ for all $w \in H$, and that $d_H^+(w) = 2$ implies $wx \in T$ (where $L = xy$). But then $H = T_4^4$ and the four implied edges yield $P(1,3,1)$ from v to x, as desired.

$P = P(1,2,1)$: As usual we may assume $d_H^+(v) = 2$, $d_H^-(v) = 1$, and likewise $d_H^-(x) = 2$, $d_H^+(x) = 1$, where $L = xy$. Let H have vertices a, b, and c with $ab, bc \in H$. If both vc and ax are in T, we have $v \to c \leftarrow b \leftarrow a \to x$. Otherwise by symmetry we may assume $cv \in T$, so $va, vb \in T$ since $d_H^+(v) = 2$. Then $cx \in T$ (else we get $u \to v \leftarrow c \leftarrow x \to y$) and $ac \in T$ (else we have $v \to b \leftarrow a \leftarrow c \to x$). Finally either $ax \in T$, giving $u \to v \leftarrow c \leftarrow a \to x$, or $xa \in T$, giving $v \to b \leftarrow a \leftarrow x \to y$.

$P = P(1,1,1,1)$: We may suppose $d_H^-(v) \leq 1$ and likewise $d_H^-(y) \leq 1$. Let H have vertices a, b and c.

If $d_H^+(v) = 3$ and $H = TT_3$, then $d_H^-(a) = 2$, say, and we may find $P' = P(1,1)$ from $\{b,c\}$ to $\{x,y\}$ giving $v \to a \leftarrow P'$ from K to L. Hence if $d_H^+(v) = 3$ we may assume $H = TC_3$, with say ab, bc and ca in H. Since $ay \in T$ gives $v \to b \leftarrow a \to y \leftarrow x$ we may assume $ya \in T$, and likewise $yb, yc \in T$. By symmetry we may now suppose $ux \in T$; then either $xa \in T$ giving $v \to c \leftarrow y \to a \leftarrow x$, or $ax \in T$, giving $u \to x \leftarrow a \to b \leftarrow y$.

Otherwise $d_H^+(v) = 2$, and likewise we may assume $d_H^+(y) = 2$. If $D_H^-(v) \cap D_H^-(y) \neq \emptyset$ we are home, so we assume av, ya, vb, yb, vc and cy are in T. Now if $ab \in T$ we get $u \to v \leftarrow a \to b \leftarrow y$, so assume $ba \in T$. Then either $cb \in T$, giving $v \to b \leftarrow c \to y \leftarrow x$, or $bc \in T$, giving $v \to c \leftarrow b \to a \leftarrow y$. $\square$

Having proved Theorem 5, we are able to easily deduce the next corollary.

COROLLARY 6. *Let C be a non-strongly oriented cycle of order n, and let T be a tournament of order $n+2$. Then T contains C provided $n \geq 14$.*

PROOF. Suppose C has a block of length $b \geq 4$. Find a vertex w with $d_T^+(w) \geq 4$ and $d_T^-(w) \geq 2$ (possible as $|T| > 10$). Now choose $K \subset D_T^+(w)$ and $L \subset D_T^-(w)$ with $|K| \geq 4$, $|L| \geq 2$ and $|K| + |L| = b+2$. (Note $b \leq n-1$.) Then if $P = P(|K|-3,\ldots,|L|-1)$ is a path obtained by removing a suitable vertex from the block of C, we may find P in $T - w$ going from K to L (using Theorems 4 and 5), and hence find C in T.

Otherwise all blocks of C have length at most 3. Let u be a vertex of C with $d_C^-(u) = 0$ and consider $C - u = P(k,\ldots,l)$, where $k, l \leq 3$. Note $P \neq P(k,1,l)$

since $n > 9$. Choose $w \in T$ with $d_T^+(w) \geq 8$, and choose K, L disjoint in $D_T^+(w)$ with $|K| = k+1$ and $|L| = l+1$. By Theorem 5 there is a copy of $P(k, \ldots, l)$ in $T - w$ running from K to L, which with w gives a copy of C in T. $\square$

Corollary 6 is probably true for all $n \geq 3$.

3. Cycles. This section contains a proof of the conjecture mentioned in the abstract. The main tool is of course Theorem 5 of §2, and the work here is devoted to removing the two unused vertices of that theorem.

We will need seven lemmas, of which Lemmas 9, 11 and 13 are the ones which will be used in the proof of the main result, Theorem 14. The first lemma is a straightforward observation.

LEMMA 7. *Let $P = x_1 \to x_2 \to \cdots \to x_k$ be a path in a tournament T, and let $Y \subset T$ be a set of vertices such that for each $y \in Y$ there is an x_i with $x_i y \in T$. Then there is a directed hamilton path in $P \cup Y$ beginning at x_1. If further for each $y \in Y$ we can find $i < j$ such that $x_i y, y x_j \in T$, then there is a directed hamilton path in $P \cup Y$ beginning at x_1 and ending at x_k.*

PROOF. It is enough to prove the lemma when $|Y| = 1$ provided the path we construct preserves the order of the x_i's. Suppose then $Y = \{y\}$. Then either there is some i with $x_i y, y x_{i+1} \in T$, when $x_1 \to x_2 \to \cdots \to x_i \to y \to x_{i+1} \to \cdots \to x_k$ is a suitable path, or else $x_k y \in T$, when $x_1 \to x_2 \to \cdots \to x_k \to y$ is a suitable path. $\square$

LEMMA 8. *Let S be a transitive subtournament of a tournament T, with $|S| \geq 4$. Let $X = T - S$ and suppose that for every $x \in X$ there are $y, z \in S$ with $xy, zx \in T$. Let m be an integer with $|X| + 3 \leq m \leq |T| - 1$. Then there is an edge uv in S and a directed path P of length m from u to v which contains X, such that $u \Rightarrow (T - P)$ and $v \Leftarrow (T - P)$.*

PROOF. Let $S = \langle w_1, \ldots, w_k \rangle$. Let $Y \subset X$ be maximal such that Y contains a directed hamilton path, say $P = x_1 \to \cdots \to x_r$ (where $r = |Y|$), with $w_i x_1$ and $x_r w_j$ in T for distinct w_i and w_j. Note $Y \neq \varnothing$ unless $X = \varnothing$ when the lemma is trivial. If $x \in X - Y$, then $Y \cup \{x\}$ has by Lemma 7 a directed hamilton path whose end vertices are either x_1 and x_r, x and x_r or x_1 and x. By the maximality of Y the first possibility cannot occur, and the others imply either $D_S^-(x) = \{w_j\}$ or $D_S^+(x) = \{w_i\}$. Hence $X - Y = X_1 \cup X_2$, where

$$X_1 = \{x \in X - Y; D_S^+(x) = \{w_i\}\} \quad \text{and} \quad X_2 = \{x \in X - Y; D_S^-(x) = \{w_j\}\}.$$

Let u be the vertex of $S - \{w_i, w_j\}$ with the smallest subscript, unless $X_1 = \varnothing$, when u is the vertex of $S - \{w_j\}$ with the smallest subscript. Let v be the vertex of $S - \{w_i, w_j\}$ with the largest subscript, unless $X_2 = \varnothing$, when v is the vertex of $S - \{w_i\}$ with the largest subscript. Then

$$Q = u \to X_1^+ \to w_i \to P \to w_j \to X_2^+ \to v$$

is a path from u to v containing X, with $u \Rightarrow (T - Q)$, $v \Leftarrow (T - Q)$ and $uv \in T$. Note Q has length at most $|X| + 3$. To obtain P, let M be $m + 1 - |Q|$ vertices of $T - Q$. Since $u \Rightarrow M$ and $M \Rightarrow v$ there is by Lemma 7 a directed hamilton path P in $Q \cup M$ from u to v. This path P satisfies the requirements of the lemma. $\square$

LEMMA 9. *Let C be a non-strongly oriented cycle of order n, and let T be a tournament of order n. If C has a block of length b and T has a transitive subtournament of order $n-b+3$, then T contains C.*

PROOF. Let S be a maximal transitive subtournament of order at least $n-b+3$. Then $|S| \geq 4$, and if $X = T - S$ then X, T and S satisfy the conditions of Lemma 8 (since S is maximal). Now $|X| + 3 = n - |S| + 3 \leq b$, so by Lemma 8 we can find $u, v \in S$ with a path P of length b from u to v, with $u \Rightarrow (T-P)$, $v \Leftarrow (T-P)$ and $uv \in T$. Since $T - P$ is transitive it contains the remainder of C, and hence T contains C. □

Note that Lemma 9 provides a proof of Grünbaum's result, where C has exactly two blocks of lengths 1 and $n-1$.

LEMMA 10. *Let P be an oriented path with at least 40 vertices and at least 3 blocks, and let T be a tournament of order $|P|$. Suppose that T contains a transitive subtournament S of order $|T|-2$. Then T contains a copy of P with initial and end vertices in S.*

PROOF. Suppose first that P has a block of length $b \geq 5$. Let $T - S = \{x_1, x_2\}$ and let X be the maximal subset of $\{x_1, x_2\}$ such that X and S satisfy the conditions of Lemma 8. If $|X| = 2$, we may by Lemma 8 find a path P_1 of length b with ends u and v such that $T - P_1 \subset S$. Since $T - P_1$ is transitive it contains $P - P_1$ and this gives us P, because $u \Rightarrow (T - P_1) \Rightarrow v$. If $X = \{x_2\}$, say, we may suppose $x_1 \Leftarrow S$. We may also write P as $R \rightarrow x \leftarrow Q$ where R and Q are paths and R, say, contains a block of length at least $b-1$. We may then find a copy R' of R containing X with its ends in S, by a similar argument to the above. But now $R' \rightarrow x_1 \Leftarrow (S-R)$ contains P, since $Q \subset (S-R)$. If $X = \varnothing$ the lemma is straightforward.

Otherwise every block of P has length at most 4. We may suppose that P can be written $Q_1 \rightarrow Q_2$, where Q_1 and Q_2 are oriented paths of lengths 19 and $|P| - 21$ respectively. Let $S = \langle w_1, \ldots, w_{|S|} \rangle$. We show that Q_1 can be found in $\langle w_1, \ldots, w_{19} \rangle \cup \{x_1\}$ with its ends in $\langle w_1, \ldots, w_{19} \rangle$. Similarly Q_2 can be found in $\langle w_{20}, \ldots, w_{|S|} \rangle \cup \{x_2\}$, which gives P. To see that Q_1 can be found as claimed, observe that two of the edges between x_1 and $\{w_9, w_{10}, w_{11}\}$ have the same direction; say without loss of generality that $w_{10}x_1, w_{11}x_1 \in T$. Write $Q_1 = R_1 \rightarrow x \leftarrow R_2$, where neither R_1 nor R_2 are devoid of edges, and let B_1 (resp. B_2) be the last (resp. first) block of R_1 (resp. R_2). By taking two suitable disjoint paths of length at most 4 from $\{w_{10}, w_{11}\}$ to $\{w_1, w_2, w_{18}, w_{19}\}$ we can construct $B_1 \rightarrow x \leftarrow B_2$, and since the ends of this path are at the extreme vertices of $\langle w_1, \ldots, w_{19} \rangle$ we get the rest of Q_1. □

LEMMA 11. *Let C be a non-strongly oriented cycle of order n containing a set E of $q \geq 43$ consecutive edges, such that E is a union of blocks of C and such that the $q-6$ consecutive edges obtained by deleting the first three and last three edges of E contain at least one block of C. Let T be a tournament of order n containing a transitive subtournament of order $q+1$. Then T contains C.*

PROOF. Let S be a transitive subtournament of order $q+1$, say $S = \langle w_0, \ldots, w_q \rangle$. We may suppose the second edge of E goes from the second vertex to the third. Let $K = \{w_0, w_1, w_2, w_3\}$ and let $L = \{w_{q-1}, w_q\}$ or $L = \{w_4, w_5\}$ according as the penultimate edge of E has the same or opposite direction as the second. Let

E_1 be E with the first and last edges removed, and let E_2 be E with the first two and last two edges removed. Let P_1 be the path obtained from C by removing E_1, and let P_2 be the path containing the edges of E_2. Then by the conditions on E, $P_1 = P(\pm 1, \ldots, 1)$ and P_2 has $q - 3 \geq 40$ vertices and at least three blocks. By Theorems 3–5 we may find P_1 in $T_1 = (T - S) \cup K \cup L$ with its ends in K and L, and to form C we need only find P_2 in $T - P_1$ with its ends in $S - K - L$. But we can achieve this by Lemma 10, because $T - P_1 = (S - K - L) \cup (T_1 - P_1)$, where $S - K - L$ is transitive of order $|P_2| - 2$ and $|T_1 - P_1| = 2$. □

Observe that Lemma 11 covers the case of alternating cycles.

We now have lemmas showing $C \subset T$ if C contains many small blocks or if C contains a very large block. Many pairs (C, T) still remain unaccounted for, but are covered by Lemma 13, which is itself based on the next lemma.

LEMMA 12. *Let S be a maximal transitive subtournament of a tournament T. Let X, Y, Z be pairwise disjoint subsets of $T - S$, and let $P = P(k, \ldots, l)$ be a path of order $|S| + |X| + |Y| + |Z|$ with at least two blocks, such that $k \geq |X| + |Z| + 6$ and $l \geq |Y| + |Z| + 6$. Suppose there are vertices $x, y \in S$ such that $x \Rightarrow X \cup (S - \{x\})$ and such that $y \Rightarrow Y \cup (S - \{x, y\})$ or $y \Leftarrow Y \cup (S - \{x, y\})$ according as P has an even or an odd number of blocks. Then there is a copy of P in $S \cup X \cup Y \cup Z$ with initial vertex x and end vertex y.*

PROOF. Notice that since $|S| + |X| + |Y| + |Z| \geq k + l + 1$ we have $|S| \geq 13$. Let $S_1 = S - \{x, y\}$, let $Y' = \{z \in Z; z \Rightarrow S_1\}$ and let $X' = Z - Y'$. Put $X \cup X' = X_1 \cup X_2 \cup X_3$, where

$$X_1 = \{w \in X \cup X'; w \Rightarrow S_1\}, \quad X_3 = \{w \in X \cup X'; w \Leftarrow S_1\}$$

and

$$X_2 = (X \cup X') - X_1 - X_3.$$

Observe that $X' \subset X_2 \cup X_3$, so $x \Rightarrow X_1$. By Lemma 8 there are vertices u_1 and v_1 in S_1 and a path P_1 in $S_1 \cup X_2$ of order $|X_2| + 4$ from u_1 to v_1, containing X_2 and such that $u_1 \Rightarrow S_2$, $v_1 \Leftarrow S_2$ and $uv \in T$, where $S_2 = (S_1 \cup X_2) - P_1$. This gives a path

$$x \to X_1^+ \to u_1 P_1 v_1 \to X_3^+ \Leftarrow S_2,$$

whose first block is of length $|X \cup X'| + 4 \leq |X| + |Z| + 4$. Note this path works even if some of the X_i's are empty, since $xu_1 \in T$ and $v_1 \Leftarrow S_2$.

If P has an even number of blocks, consider some $z \in Y'$. If $zy \in T$, then either $\langle x, z, y, \ldots \rangle$ or $\langle z, x, y, \ldots \rangle$ is a transitive tournament containing S. Hence by the maximality of S, $yz \in T$, and so $y \Rightarrow Y'$. Thus we can repeat the argument of the previous paragraph with x, X and S_1 replaced by y, Y and S_2, and since $y \Rightarrow Y_1$ we have the path

$$y \to Y_1^+ \to u_2 P_2 v_2 \to Y_3^+ \Leftarrow S_3,$$

where $S_3 = (S_2 \cup Y_2) - P_2$. The first block of this path has length $|Y \cup Y'| + 4 \leq |Y| + |Z| + 4$, and furthermore $|S_3| = |S_1| - 8 = |S| - 10$.

On the other hand, if P has an odd number of blocks, then partition $Y \cup Y'$ as $Y_1 \cup Y_2 \cup Y_3$, where

$$Y_1 = \{w \in Y \cup Y'; w \Leftarrow S_2\}, \quad Y_3 = \{w \in Y \cup Y'; w \Rightarrow S_2\}$$

and

$$Y_2 = (Y \cup Y') - Y_1 - Y_3.$$

Since $Y' \subset Y_3$ we have $y \Leftarrow Y_1$, and so in a similar manner a path

$$S_3 \Leftarrow Y_3^+ \to u_2P_2v_2 \to Y_1^+ \to y$$

can be constructed, the last block being of length $|Y \cup Y'| + 4$. Notice $|S_3| = |S| - 10 > 0$.

Suppose first that $|P| \geq k + l + 3$. Choose $R_1 \subset S_3$ with $|R_1| = k - |X \cup X'| - 4$ and $R_2 \subset S_3 - R_1$ with $|R_2| = l - |Y \cup Y'| - 4$. Let $R = S_3 - R_1 - R_2$. Note $|R| = |S_3| - |R_1| - |R_2| = |P| - 2 - k - l \geq 1$. By Lemma 7, $u_iP_iv_i$ together with R_i contains a directed hamilton path Q_i from u_i to v_i, because $u_i \Rightarrow R_i \Rightarrow v_i$. We now have

$$x \to X_1^+ \to Q_1^+ \to X_3^+ \Leftarrow R \Rightarrow Y_3^- \leftarrow Q_2^- \leftarrow Y_1^- \leftarrow y,$$

or

$$x \to X_1^+ \to Q_1^+ \to X_3^+ \Leftarrow R \Leftarrow Y_3^+ \to Q_2^+ \to Y_1^+ \to y,$$

as appropriate, from x to y, with first and last blocks lengths k and l respectively. Since R is transitive, we have P from x to y.

Otherwise $|P| \leq k + l + 2$, so $P = P(k,1,l)$ or $P = P(k,l)$. Suppose $P = P(k,1,l)$. Consider the edge from the end of the directed path $v_1 \to X_3^+$ to the beginning of the directed path $Y_3^+ \to u_2$. If this goes from the latter to the former (in particular if $Y_3 = \emptyset$, since $u_2 \in S_2$ and both $v_1 \Leftarrow S_2$ and $X_3 \Leftarrow S_2$) form Q_1 and Q_2 as before to get

$$x \to X_1^+ \to Q_1^+ \to X_3^+ \leftarrow Y_3^+ \to Q_2^+ \to Y_1^+ \to y,$$

which is the desired $P(k,1,l)$ from x to y. Otherwise this edge goes from $v_1 \to X_3^+$ to a vertex $w \in Y_3 \neq \emptyset$. Now choose $R_1 \subset S_3$ with $|R_1| = k - |X \cup X'| - 6$ and $R_2 \subset S_3 - R_1$ with $|R_2| = l - |Y \cup Y'| - 3$. Form Q_1 and Q_2 as before. Notice that

$$|S_3 - R_1 - R_2| = |S| - |P| + |X| + |Y| + |Z| + 1 = 1,$$

so put $S_3 - R_1 - R_2 = \{r\}$; note $wr \in T$ since $w \in Y_3$. We then have

$$x \to X_1^+ \to Q_1^+ \to X_3^+ \to w \to r \leftarrow (Y_3 - w)^+ \to Q_2^+ \to Y_1^+ \to y,$$

which is the $P(k,1,l)$ we want.

Finally, suppose P has exactly two blocks; then $|S| + |X| + |Y| + |Z| = k + l + 1$. Consider the connecting edge between the end vertices of the paths $v_1 \to X_3^+$ and $v_2 \to Y_3^+$; by symmetry we may suppose it goes from the latter to the former. Then choose R_1 and R_2 as before but with $|R_1| = k - |X \cup X'| - 4$ and $|R_2| = l - |Y \cup Y'| - 5$. (Of course, if the connecting edge goes the other way we reduce R_1 instead of R_2.) Notice now $R = \emptyset$. Constructing Q_1 and Q_2 as before gives

$$x \to X_1^+ \to Q_1^+ \to X_3^+ \leftarrow Y_3^- \leftarrow Q_2^- \leftarrow Y_1^- \leftarrow y,$$

which is $P(k,l)$ from x to y. □

We are now ready to prove the main lemma.

LEMMA 13. *Let C be a non-strongly oriented cycle of order n, let T be a tournament of order n, and let S be a maximal transitive subtournament of T. Suppose that C contains distinct blocks B_1 and B_2 of lengths b_1 and b_2 each at least 18, such that the distance d between B_1 and B_2 (measured one of the two ways on C) satisfies $d \leq |S| - 33$ and $d + b_1 + b_2 \geq |S| + 10$. (We allow $d = 0$ if B_1 and B_2 are adjacent.) Then T contains C.*

PROOF. Let $P_1 = P(m, \ldots)$ be the path of length d in C from B_1 to B_2, and let $P_2 = P(q, \ldots)$ be the path of length $n - d - b_1 - b_2$ from B_1 to B_2. (We may assume, by reversing all arcs in C and T if necessary, that both $m \geq 0$ and $q \geq 0$; either or both of P_1 and P_2 may have length zero.) Let x, y be vertices of S defined by $S = \langle x, \ldots, y\rangle$ if B_1 and B_2 have the same direction on C (that is, P_1 and P_2 have an odd number of blocks) and $S = \langle x, y, \ldots\rangle$ otherwise. Let $H = T - S$. The aim of the proof is to find a path from x to y via H containing as much of P_2, B_1 and B_2 as possible (using Theorems 3–5), and then to find the rest of C as a path from x to y in S and the other remaining vertices (using Lemma 12).

The proof falls into three cases depending on the values of $d_H^+(x)$ and $d_H^+(y)$. In what follows the symbols d^1 and d^2 are equivalent to d^+ and d^- respectively if B_1 and B_2 have the same direction in C, and equivalent to d^- and d^+ respectively if B_1 and B_2 have opposite directions.

Case (a): $d_H^-(x) \geq 6,\ d_H^1(y) \geq 6$. Suppose there are integers k and l such that we can find paths $P(b_1 - k, m, \ldots, b_2 - l)$ from x to y and $P(-k, q, \ldots, l)$ from x to y whose intersection is just $\{x, y\}$. Then we will have the desired copy of C. To find such paths, first choose k and l by the following procedure (which essentially finds the largest feasible values for k and l).

Choose $K_1 \subset D_H^-(x),\ |K_1| = 4$, and $L_1 \subset D_H^1(y) - K_1,\ |L_1| = 2$. Now choose $K_2 \subset D_H^-(x) - K_1 - L_1$ and $L_2 \subset D_H^1(y) - K_1 - L_1$ with $K_2 \cap L_2 = \varnothing$ and $|K_2| + |L_2|$ as large as possible, subject to

(i) $|K_2| \leq b_1 - 18$,

(ii) $|L_2| \leq b_2 - 18$, and

(iii) $|K_2| + |L_2| \leq d + b_1 + b_2 - |S| - 5$.

Put $K = K_1 \cup K_2,\ L = L_1 \cup L_2,\ k = |K| - 2$ and $l = |L|$. Then $k \geq 2$ and $l \geq 2$, and (i)–(iii) are equivalent to

(i) $k \leq b_1 - 16$,

(ii) $l \leq b_2 - 16$, and

(iii) $k + l \leq d + b_1 + b_2 - |S| - 1$.

Inequality (iii) is the same as

$$k + l \leq n - |P(q, \ldots)| - |S|,$$

whence $|P(-(k-1), q, \ldots, l-1)| \leq n - |S| - 2 = |H| - 2$. Let $P^* = P(-(k-1), q, \ldots, l-1)$. If $P^* \neq P(-(k-1), l-1)$, then $|P^*| \geq k + l$ and we may choose a tournament $T^* \subset H$ of order $|P^*| + 2$ containing K and L; otherwise if $P^* = P(-(k-1), l-1)$ put $T^* = K \cup L$, whence T^* has order $|P^*| + 3$.

By Theorems 3–5 we may find a copy of P^* in T^* beginning in K and ending in L. Since $K \subset D_H^-(x)$ and $L \subset D_H^1(y)$, this gives a copy of $P(-k, q, \ldots, l)$ from x to y. What remains to be done is to find a copy of $P(b_1 - k, m, \ldots, b_2 - l)$ from x to y in $T - P^*$. Let $Z = T^* - P^*$; then $2 \leq |Z| \leq 3$. Let $W = T - T^* - S$; thus $T - P^* = S \cup Z \cup W$. Observe $|W| + |Z| = |T| - |P^*| - |S| = d + b_1 + b_2 + 1 - k - l - |S|$.

We shall now find a partition of W into $X \cup Y$, with $X \subset D^+(x)$, $Y \subset D^2(y)$, $b_1 - k \geq |X| + |Z| + 6$ and $b_2 - l \geq |Y| + |Z| + 6$. Then Lemma 12 shows that the desired copy of $P(b_1 - k, m, \ldots, b_2 - l)$ exists.

Suppose first equality holds in (iii). Then $|W| + |Z| = 2$, so $|Z| = 2$ and $W = X = Y = \emptyset$. Note $b_1 - k \geq 16 \geq |Z| + 6$, $b_2 - l \geq 16 \geq |Z| + 6$. Now suppose that inequality (iii) is strict. Observe that strict inequality in (i) now means that $D_H^-(x) \subset K_2 + K_1 + L_2 + L_1 \subset T^*$ (or else K_2 could have been made larger) so $W \subset D^+(x)$. Likewise strict inequality in (ii) implies $W \subset D^2(y)$. Now $d \leq |S| - 33$ means

$$|W| \leq b_1 + b_2 - k - l - 32 - |Z|,$$

so we can partition W as $X \cup Y$ with

$$|X| \leq b_1 - k - 16 \quad \text{and} \quad |Y| \leq b_2 - l - 16.$$

If $X \neq \emptyset$ then inequality (i) is strict, so $X \subset D^+(x)$, and likewise $Y \subset D^2(y)$ if $Y \neq \emptyset$. Since $|Z| + 6 \leq 16$, this is the partition of W we seek.

Case (b): *$d_H^-(x) \leq 5$ and $d_H^1(y) \geq 6$, or $d_H^-(x) \geq 6$ and $d_H^1(y) \leq 5$.* We prove this case on the assumption that the first pair of inequalities hold. It will then be clear how the proof should be modified if the second pair of inequalities hold. Choose $L \subset D_H^1(y)$ as large as possible, subject to

(i) $l \leq b_2 - 16$, and

(ii) $l \leq d + b_1 + b_2 - |S| - 7$,

where $l = |L|$. Then $l \geq 2$. (However, if $P_2 = P(q, \ldots)$ has order 1 or 2, so B_1 and B_2 are adjacent in C or separated by one edge, we choose $L \subset D_H^1(y) \cap D_H^+(x)$, again subject to (i) and (ii). Note $l \geq 1$ since $d_H^+(x) \geq |H| - 5$, and in this case alone we allow $l = 1$.)

Let $P^* = P(q - 1, \ldots, l - 1)$, or $P^* = P(l - 1)$ if $|P_2| = 1$ or 2. We choose $K \subset D_H^+(x)$ and T^* as follows. If P_2 has order 1 or 2 we put $T^* = K = L$. It is possible here that $l = 1$ and T^* is a single vertex. Otherwise choose $K \subset D_H^+(x) - L$ with $|K| = q + 2$. This is possible because $|H| = n - |S|$ and $n \geq d + b_1 + b_2 + q$, so by (ii)

$$d_H^+(x) \geq |H| - 5 \geq d + b_1 + b_2 + q - |S| - 5 \geq l + q + 2.$$

If P^* has two blocks, that is $P^* = P(q-1, l-1)$, put $T^* = K \cup L$, so $|T^*| = |P^*| + 3$. Otherwise choose $T^* \subset H$ with $K \subset T^*$, $L \subset T^*$ and $|T^*| = |P^*| + 2$; again this is possible because by (ii),

$$|P^*| + 2 = n - d - b_1 - b_2 + l + 1 \leq n - |S| - 6 < |H|.$$

Using Theorems 3–5 we find P^* in T^* beginning in K and ending in L. Because $K \subset D_H^+(x)$ and $L \subset D_H^1(y)$ we then have $P(q, \ldots, l)$ from x to y, and to complete C we will find $P(b_1, m, \ldots, b_2 - l)$ in $T - P^*$ beginning at x and ending at y. Put $T - P^* = S \cup W \cup Z'$, where $W = T - T^* - S - D_H^-(x)$ and $Z' = T^* + D_H^-(x) - P^*$. Note $|Z'| \leq |D_H^-(x)| + |T^* - P^*| \leq 8$. Once more we are home by Lemma 12 if we partition $W \cup Z'$ as $X \cup Y \cup Z$ with $X \subset D^+(x)$, $Y \subset D^2(y)$, $b_1 \geq |X| + |Z| + 6$ and $b_2 - l \geq |Y| + |Z| + 6$.

Now $|W| + |Z'| = n - |P^*| - |S| = d + b_1 + b_2 + 1 - l - |S|$. If equality holds in (ii), then $|W| + |Z'| = 8$, so it is satisfactory to put $Z = W \cup Z'$ and $X = Y = \emptyset$, because $b_1 \geq 18$ and $b_2 - l \geq 16$. If inequality is strict in (ii), put $Z = Z'$ and

$W = X \cup Y$, where $|X| \leq b_1 - 16$ and $|Y| \leq b_2 - l - 16$. This is possible because $|W| \leq b_1 + b_2 - l - 32$. Observe that $|Z| + 6 \leq 16$ since $|Z'| \leq 8$, and that $X \subset W \subset D^+(x)$ by definition of W. Finally, if $Y \neq \emptyset$ then inequality is strict in (i) also, and hence $(D^1_H(y) \cap D^+_H(x)) \subset L \subset T^*$ (otherwise L would have been chosen larger). But $W \subset (T - T^*) \cap D^+_H(x)$ so $Y \subset W \subset D^2(y)$, as desired.

Case (c): $d^-_H(x) \leq 5$, $d^1_H(y) \leq 5$. Let $T^* = D^+_H(x) \cap D^2_H(y)$; then $|T^*| \geq |H| - 10$. If $P_2 = P(q, \ldots) = P(q, \ldots, r)$ has order at least 3, let $P^* = P(q-1, \ldots, r-1)$ (or if $P_2 = P(q)$ put $P^* = P(q-2)$). Then $|P^*| \geq 1$, and

$$|P^*| = |P_2| - 2 = n - d - b_1 - b_2 - 1 \leq n - |S| - 11 = |H| - 11 \leq |T^*| - 1.$$

Hence by Corollary 2 we may find P^* in T^*, giving P_2 between x and y. If P_2 has order 2 we already have P_2 between x and y, since $xy \in S$; put $P^* = \emptyset$ in this case. Thus if $|P_2| \geq 2$ we need only find $P(b_1, m, \ldots, b_2)$ from x to y in $T - P^*$. If P_2 is a single vertex, it is enough to find $P(b_1, m, \ldots, b_2 - 1)$ in T from x to y, since $xy \in S$; put $P^* = \emptyset$ in this case also. Let $b = b_2$ if $|P_2| \geq 2$ and $b = b_2 - 1$ if $|P_2| = 1$.

Let $Z = D^-_H(x) \cup D^1_H(y)$; then $|Z| \leq 10$. Let $W = T - Z - S - P^*$. We must find $P(b_1, m, \ldots, b)$ in $T - P^*$ from x to y. Since $T - P^* = W \cup Z \cup S$, we may apply Lemma 12 provided we partition W as $X \cup Y$, where $b_1 \geq |X| + |Z| + 6$ and $b \geq |Y| + |Z| + 6$. (Observe that, by definition, $W \subset D^+_H(x) \cap D^2_H(y)$, so $X \subset D^+(x)$ and $Y \subset D^2(y)$.) To do this, just choose X and Y with $|X| \leq b_1 - 16$ and $|Y| \leq b - 16$. This is possible since

$$|W| + |Z| = d + b_1 + b + 1 - |S| \leq b_1 + b - 32. \quad \square$$

From Lemmas 9, 11 and 13 follows the main result.

THEOREM 14. *Let C be a non-strongly oriented cycle of order n. Let T be a tournament of order n. If T contains a transitive subtournament of order* 129 (*in particular if $n \geq 2^{128}$*), *then T contains C.*

PROOF. Label the edges of C consecutively $e(0), e(1), \ldots, e(n-1)$ so that $e(115)$ and $e(116)$ are in different blocks. Let S be a maximal transitive subtournament of T with $|S| \geq 129$, and let $m = |S| - 11 \geq 118$. For $0 \leq j \leq 115$ we may assume that $e(j)$ and $e(m+j)$ are in different blocks (addition is mod n), else as $m + j \geq 118$ they are both in a block of length at least $n - |S| + 11$, whence by Lemma 9 T contains C.

Now suppose that for some t, $0 \leq t \leq 4$, the edges $\{e(23t + j);\ 0 \leq j \leq 21\}$ are in the same block B_1 of length $b_1 \geq 22$ and the edges $\{e(23t + j + m);\ 0 \leq j \leq 21\}$ are in the same block B_2 of length $b_2 \geq 22$. Then $B_1 \neq B_2$, and if the distance between B_1 and B_2 (measured along the path of C containing $e(115)$) is d we have

$$d \leq 23t + m - 1 - (23t + 21) \leq |S| - 33$$

and $d + b_1 + b_2 \geq 23t + m + 21 - (23t - 1) = |S| + 11$. Then by Lemma 13 C is in T.

Otherwise, either there are at least three values of t, $0 \leq t \leq 4$, for which the set $\{e(23t + j);\ 0 \leq j \leq 21\}$ contains edges from different blocks, or there are at least three values of t for which the set $\{e(23t + j + m);\ 0 \leq j \leq 21\}$ contains edges from different blocks. Construct a set E of edges as follows. In the first case, let $e(k)$ be the first edge in a different block from $e(0)$, and put $E = \{e(k), \ldots, e(115)\}$. In the second let k be the largest integer with $k \leq 112 + m$ and with $e(k)$ and $e(113+m)$ in different blocks, and put $E = \{e(116), \ldots, e(k)\}$. Then $50 \leq |E| \leq |S| - 14$, and E satisfies the conditions of Lemma 11, so T contains C. □

COROLLARY 15. *Let T be a tournament of order $n \geq 2^{128} + 1$. Then T is pancyclic; that is, T contains every non-strongly oriented cycle C with $3 \leq |C| \leq n$.*

PROOF. If $|C| \leq 13$, then $C \subset TT_{13} \subset T$. If $14 \leq |C| \leq n - 2$, then $C \subset T$ by Corollary 6. Otherwise $n - 1 \leq |C| \leq n$ and $C \subset T$ by Theorem 14. □

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